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Statistics of the Geomagnetic Secular Variation for the Past 5 m.y.

C. G. CONSTABLE AND R. L. PARKER

Institute of Geophysics and Planetary Physics, Scripps Institution of Oceanography
University of California, San Diego

A new statistical model is proposed for the geomagnetic secular variation over the past 5 m.y. Unlike previous models, which have concentrated upon particular kinds of paleomagnetic observables, such as VGP or field direction, the new model provides a general probability density function from which the statistical distribution of any set of paleomagnetic measurements can be deduced. The spatial power spectrum of the present-day nondipole field is consistent with a white source near the core-mantle boundary with Gaussian distribution. After a suitable scaling, the spherical harmonic coefficients may be regarded as statistical samples from a single giant Gaussian process; this is our model of the nondipole field. Assuming that this characterization holds for the fields of the past, we can combine it with an arbitrary statistical description of the dipole. We compute the corresponding probability density functions and cumulative distribution functions for declination and inclination that would be observed at any site on the surface of the Earth. Global paleomagnetic data spanning the past 5 m.y. are used to constrain the free parameters of the model, i.e., those giving the dipole part of the field. The final model has these properties: (1) with two exceptions, each Gauss coefficient is independently normally distributed with zero mean and standard deviation for the nondipole terms commensurate with a white source at the core surface; (2) the exceptions are the axial dipole g_1^0 and axial quadrupole g_2^0 terms; the axial dipole distribution is bimodal and symmetric, resembling a combination of two normal distributions with centers close to the present-day value and its sign-reversed counterpart; (3) the standard deviations of the nonaxial dipole terms g_1^1 and h_1^1 and of the magnitude of the axial dipole are all about 10% of the present-day g_1^0 component; and (4) the axial quadrupole reverses sign with the axial dipole and has a mean magnitude of 6% of its mean magnitude. The advantage of a model specified in terms of the spherical harmonic coefficients is that it is a complete statistical description of the geomagnetic field, capable of simultaneously satisfying many known properties of the field. Predictions about any measured field elements may be made to see if they satisfy the available data.

INTRODUCTION

It is now generally accepted that the main geomagnetic field is generated by some sort of dynamo process in the Earth's core (see, for example, *Moffatt* [1978]). The field is by no means constant; changes on time scales from 10 years to 10,000 years are classified as secular variations. Such fluctuations are consequences of the nonsteady nature of the field-producing mechanism within the core. More rapid variations, if they exist, cannot be detected because of screening due to mantle conductivity and interference from fluctuations of external origin. At the longest periods, dissociation of mere secular variation from the actual reversal of the main field is difficult, and it is quite possible the distinction is artificial. Unfortunately, the theory of the geodynamo is not currently in an advanced enough state to make more than the simplest predictions about the paleofield variation [e.g., *Merrill and McElhinny*, 1983, chapter 9]. When a system is as complex in space and time as the geomagnetic field, it is a natural and efficient strategy to call upon some kind of statistical characterization of it to uncover general properties, rather than to attempt a description of the behavior of the system at every instant and at all points. Previous statistical descriptions of the paleomagnetic field have centered around the use of *Fisher* [1953] statistics and attempts to describe the observed variation in dispersion of geomagnetic field directions (from paleomagnetic data) as a function of latitude [*Creer et al.*, 1959; *Creer*, 1962; *Cox*, 1962, 1970; *Baag and Helsley*, 1974; *McElhinny and Merrill*, 1975; *Harrison*, 1980; *McFadden and McElhinny*, 1984]. Allan Cox was one of the major contributors to these models. His model C [*Cox*, 1962] attributed the observed dispersion to dipole wobble and nondipole components based on parameters derived from the present field. Subsequently, [*Cox* 1970] derived a means of converting the angular dispersion of field direction to that of pole positions and conversely. This was used to try and seperate contributions to the observed dispersion due to dipole wobble or due to the nondipole field. The most successful of these statistical descriptions to date has been the so-called model F of *McFadden and McElhinny* [1984]. In this model the average intensity of nondipole components at any latitude is assumed proportional to the intensity of the dipole field, and the angular distribution of VGP directions is given axial symmetry. Model F provides a good fit to the variation in VGP angular dispersion as a function of latitude but makes no predictions about other paleofield properties such as intensity. The aim of this work is to provide a statistically coherent description of the field variation by combining the properties of the present-day field and of paleomagnetic measurements. The model is specified in terms of the spherical harmonic coefficients describing the geomagnetic field and in principle is capable of providing the probability density function of any measured quantity.

Broadly speaking, two kinds of data describe the main geomagnetic field: the modern data, with their excellent geographical coverage of the Earth, and the paleosecular variation data which span long time periods but, in general, have poor global coverage. Each one supplies a component of our model. The modern data provide a good estimate of the spatial power spectrum of the geomagnetic field; it is well known that the nondipole part of the field is consistent with a white source near the core-mantle boundary. By suitable scaling of the spherical harmonic coefficients we find that they can be regarded as statistical samples from a single giant Gaussian distribution. Assuming a kind of uniformitarianism, we take the giant Gaussian process to be the description of the nondipole paleomagnetic field. However, the modern data are

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less helpful in describing the dipole, which obviously does not fit into the general pattern of the higher harmonics. We turn to the paleodata for constraints upon the dipole behavior.

The most accurately measured paleodata are inclination and declination. We have analyzed a global data set based on a compilation by *Lee* [1983] spanning the last 5 m.y.; this period should be long enough to provide a reasonably complete sample of the secular variation but is short enough to avoid complications of plate tectonic movements. Certain empirical cumulative distribution functions are estimated from these data. As a first attempt, we introduce into the model a number of free parameters to represent the variation of the dipole field. We find, however, that the cumulative distributions derived from the data and those of the simple model are difficult to reconcile without introducing another component: a nonzero quadrupole term greatly improves the performance of the model, in general agreement with the findings of many paleomagnetic studies.

SHORT-TERM DATA AND NONDIPOLE FIELD STATISTICS

To summarize the modern geomagnetic field, we take the GSFC980 spherical harmonic model of *Langel et al.* [1980], derived from Magsat data. The magnetic field of internal origin is expressed as the gradient of a potential Ψ completely specified by an infinite sum of spherical harmonics:

$$\Psi = a \sum_{l=1}^{\infty} \sum_{m=0}^{l} \left(\frac{a}{r}\right)^{l+1} (g_l^m \cos m\phi + h_l^m \sin m\phi) P_l^m(\cos\theta) \qquad (1)$$

Conventionally, g_l^m and h_l^m are known as the Schmidt partially normalized Gauss coefficients. Appendix 1 gives the relationships between fully and partially normalized spherical harmonic representations both of which are employed in this paper.

Previous secular variation studies have been seriously hampered by their inability to treat the nondipole components of the field in a quantitative manner consistent with (1). We propose a way out of the difficulty based upon a simple observation about the present-day field: the nondipole terms can be described by a zero-mean Gaussian process, and it is plausible to assume this property has held for all time.

Mauersberger [1956] first defined a kind of geomagnetic power spectrum based on the spherical harmonic expansion. Subsequently, other workers have used various forms for the spectrum differing in the weighting as a function of degree. We shall use the form devised by *Lowes* [1974]. For each spherical harmonic of degree l, the power at radius r is R_l,

$$R_l(r) = \left(\frac{a}{r}\right)^{2(l+2)} (l+1) \sum_{m=0}^{l} \{ (g_l^m)^2 + (h_l^m)^2 \} \qquad (2)$$

Equivalently,

$$R_l = \langle \mathbf{B}_l \cdot \mathbf{B}_l \rangle$$

where $\mathbf{B}_l$ is the magnetic field associated with degree l in the expansion and $\langle\ \rangle$ denote the average over the surface of a sphere of radius r.

At the Earth's surface, $r = a$, the spectrum drops exponentially with a slope indicating a white source approximately at the core's surface (see Figure 1). Lowes gives

$$R_l \simeq 4.0\times10^3\,(4.5)^{-l}\,(\mu\text{T})^2$$

The deviations from this law for $l > 8$ are thought to represent power from crustal magnetization. The spectrum has subsequently been refined, through the use of the more complete Magsat data set [*Langel and Estes* 1982]. They estimate that the spectrum is consistent with a white source 174 km below the seismic core-mantle boundary.

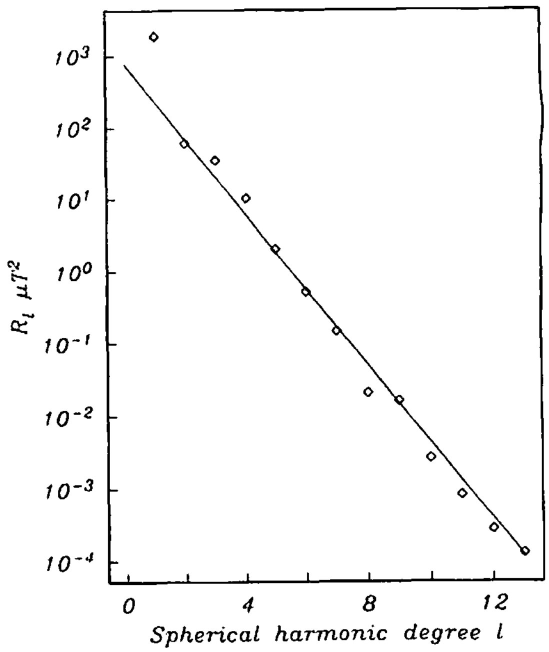

Fig. 1. The geomagnetic spectrum at the Earth's surface, computed from GSFC980 magnetic field model. The fitted straight line is for a white source at the surface of the core for terms from $l = 2$ to 8.

If the spectrum R_l were precisely the same throughout geologic time, then secular variation would just be reflected by a rearrangement within the individual g_l^m and h_l^m in (2) while the sum remained constant. This implies a kind of $2l+1$-dimensional Fisherian distribution for the Gauss coefficients. However, exactly constant power in each spherical harmonic degree contradicts observation: the time derivatives of modern spherical harmonic models show us that continual small changes in R_l are undoubtedly occurring. We therefore choose a description in which the power is also a random variable. We put forward a model that treats the individual spherical harmonic coefficients as independent, normally distributed random variables. This simple model or a similar assumption has been used by other workers as a statistical description of the secular variation (see, for example, *Gubbins* [1983] and *Eckhardt* [1984]) and has some very attractive features, not the least of which is that the model is amenable to testing, at least for the present-day field.

We use the Lowes spectrum as a guide in the construction of our statistical model. For the nondipole terms we choose a model spectrum that is exactly flat at the surface of the core:

$$\mathrm{E}\{R_l\} = \mathrm{E}\left\{ \sum_{m=0}^{l} (l+1)\,[(g_l^m)^2 + (h_l^m)^2] \right\}$$

$$= (c/a)^{2l}\,\alpha^2, \qquad l \geq 2$$

where E{ } is the expectation of the parameter in the bracket $c/a = 0.547$ is the ratio of the core radius to that of the Earth, and $\alpha = 27.7\,\mu$T is a fitted parameter; the line in Figure 1 conforms to this equation. In addition, we assume that within each degree l, g_l^m and h_l^m are independent, identically distributed, normal random variables of zero mean. If $\mathrm{Var}(h_l^m) = \mathrm{Var}(g_l^m) = \sigma_l^2$, then

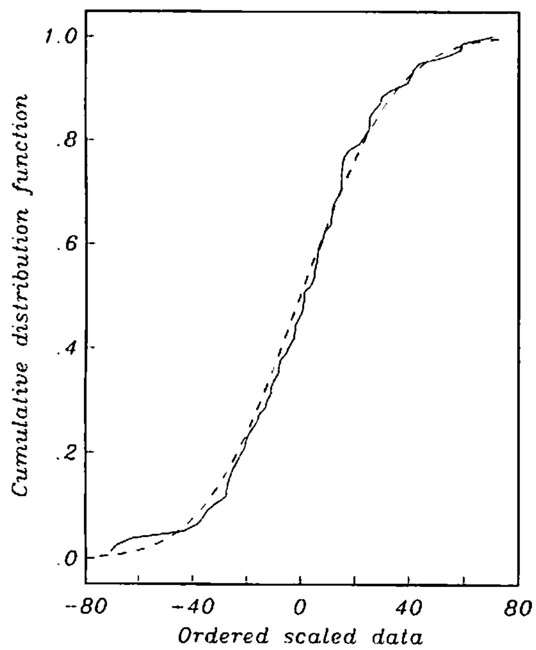

Fig. 2. Empirical cdf for the GSFC980 Gauss coefficients scaled in the manner described in the text. The dashed curve shows the theoretical curve expected if the scaled coefficients correspond to a Gaussian distribution.

$$\sigma_l^2 = \frac{(c/a)^{2l}\alpha^2}{(l+1)(2l+1)}$$

If we define scaled Gauss coefficients

$$\tilde{g}_l^m = v_l g_l^m$$
$$\tilde{h}_l^m = v_l h_l^m$$

where $v_l^2=(l+1)(2l+1)(a/c)^{2l}$, then these coefficients (which completely characterize the nondipole field for statistical purposes) are just independent samples of a single zero-mean Gaussian process with variance α^2. Even though there is, as we noted, some evidence to indicate that the level at which the spectrum is flat lies somewhat below the core-mantle boundary, we prefer to characterize the spectrum in terms of the known physical constants c and a. We do not believe the true level is very well constrained or necessarily has great physical significance. In any case, the method used here in fitting the data yields an excellent fit to both the spectrum (see the straight-line fit on Figure 1) and, as we will see, the Gaussian distributional form for the nondipole field. If the level at which the spectrum is flat is changed, this affects only the variance α^2 of the scaled Gaussian distribution.

The spherical harmonic field coefficients for 1980 (GSFC980) data for degree $l = 2$ through to $l = 8$ or higher were used to test the validity of the Gaussian model in the scaled coefficients. Figure 2 shows the empirical cumulative distribution versus the theoretical normal curve; the ordinate represents the fraction of the data with values less than or equal to the abscissa. The agreement between the two curves is impressive. While there is no means of determining a unique statistical model from these data, it is possible to test whether a data sample is consistent with an assumed distribution. The Kolmogorov-Smirnov test (see, for example, *Massey* [1951] and *Kendall and Stuart* [1979]) is a powerful means of determining probabilistic bounds within which samples from a particular theoretical distribution function should lie; one virtue of it is that the test itself is independent of the underlying theoretical distribution function (although of course this must be known to compute the discrepancy statistic d_N). We find that the data of Figure 2 are consistent with an underlying normal distribution for the nondipole field; the maximum discrepancy from the theoretical normal distribution is d_N=.061, where N=77. The probability of such a deviation or larger arising by chance is, from the standard tables, 0.9988. However, there is reason to believe that the dipole part of the field is different from the rest. Figure 1 shows that the dipole part of the spectrum does not conform to the least squares line fit to the nondipole part. The major contributor to this is the g_1^0 axial part of the dipole. In the scaled distribution the $\tilde{g}_1^0$ term lies at 4.85 standard deviations from the zero mean. For a random sample from a distribution this has a probability of $< 3.3 \times 10^{-5}$, hardly what one would expect for a typical sample from the distribution. One might suspect that just the the axial part of the dipole is anomalous, and this appears to be confirmed by the size of the $\tilde{g}_1^1$ and $\tilde{h}_1^1$ terms, both of which lie within one standard deviation of the zero mean.

As a description of the nondipole field, our model has simplicity and economy; only one number, the variance, needs to be determined empirically. It seems extremely unlikely that such a simple description is an accident and pertains to today's field alone. Rather, it is a reasonable working hypothesis that the model accurately provides the statistics of the nondipole part of the paleofield as well. Under this assumption we can compute the statistics of the "noise" contributed to paleomagnetic measurements due to nondipole sources. In Appendix 2 we show (1) that the components of the nondipole field at the Earth's surface are also independent Gaussian variables with zero means, (2) that the variances of the horizontal components locally are equal, and (3) that the variance of the vertical component is not the same as the other two. The inequality of the vertical and horizontal variances makes it evident that the distribution of directions for the nondipole part of the field is not Fisherian, something entirely consistent with observation.

A STATISTICAL DISTRIBUTION FOR THE DIPOLE FIELD

For now we shall assume that the giant Gaussian of the previous section is a complete description for the statistical variations in the nondipole field. What is an appropriate model for the dipole field? This is not a trivial matter, and indeed, a large part of the science of paleomagnetism is devoted to exactly this question: How does the dipole of the geomagnetic field behave? Continuing with the policy adopted for the nondipole field, we will ignore questions of development with time and concentrate instead on the statistical distribution of the dipole field amongst its various possible states.

In all of the following models (with one exception) it will be assumed that the dipole part of the field is statistically independent of the nondipole part. This is clearly not true in general: the complex evolution of secular variation must be controlled by the geodynamo, in which the dipole and nondipole variations are presumably inextricably linked. For example, eastward and westward drift of features observed in the field elements suggest that there is transfer of energy between different order terms within the same degree. However, in developing a statistical model for the secular variation we will make use of paleomagnetic data, which provides a sporadic sampling of the process throughout time, on time scales much longer than that typically exhibited by the westward drift. The assumption of independence is then like proposing there exists no requirement for the quadrupole term to be small whenever the dipole term is large, or similar relationships between other spherical harmonic coefficients. Assuming that the data are not serially correlated (certainly a safer assumption for lava flows than for sediments), the independence assumption should not be a serious problem.

The simplest idea would be to treat the dipole as just another part of the giant Gaussian distribution. Then the three components of the dipole g_1^0, g_1^1, and h_1^1 would exhibit identical behavior. This is completely incompatible with what is known of field component distributions from paleomagnetic data [*Cox*, 1970] in which the axial term, that is, g_1^0, has been dominant throughout geologic time (except during the brief periods of actual reversal). We therefore rule out this model.

The least perturbation to an entirely isotropic dipole model is one that ascribes to the dipole variations the same statistical behavior as the nondipole field but, during times not associated with reversal, superimposes a steady axial part to account for the ascendancy of the g_1^0 term. This seems a reasonable approach in view of the observation that the g_1^1 and h_1^1 terms are compatible with the giant Gaussian model of nondipole variations. Thus the statistical distribution for the g_1^0 term is proposed to be bimodal and symmetric, a combination of two normal distributions with centers at the present-day value and its sign reversed counterpart. We take this opportunity to introduce some new terminology specifying the magnitude of the zonal terms of the geomagnetic field. Let $\gamma_l^0 = |g_l^0|$ for l=1,2, $\cdots$; then the statistical distribution for γ_1^0 will be closely approximated by a Gaussian distribution centered on $|g_1^0|$. (The standard deviation computed from the present-day spectrum for the dipole parts of the field is about 20% of the present axial dipole magnitude; thus the area under the normal probability density function (pdf) that is truncated by taking the absolute value will be very small.) Then at times not associated with reversal the magnetic field components may be written as

$$B_r' = B_r + f$$

$$B_\theta' = B_\theta + e$$

$$B_\phi' = B_\phi$$

where $f = 2\gamma_1^0 \sin\lambda$, $e = \gamma_1^0 \cos\lambda$ are the mean contributions at a site at latitude λ and, as we mentioned earlier, B_r, B_θ, and B_ϕ, the random components, are Gaussian, zero mean, and independent with variances derived from the spectrum at the core. We can compute the pdfs for the commonly measured elements of the geomagnetic field by performing the necessary integrals. The techniques used for performing these calculations are laid out in Appendix 3 for a general form for the axial dipole distribution. Numerical integration of those pdfs yield cumulative distribution functions (cdfs) for comparison with the empirical cdfs obtained for the data. To test this model requires a paleomagnetic data base and the development of some suitable parameters for comparison. These are the subject of the next section

A PALEOMAGNETIC DATA SET AND ITS DISTRIBUTION

Figure 3 summarizes the primary paleomagnetic data that we have used to test the model. These are taken from the subset of the global data set compiled by *Lee* [1983] for which individual site measurements are available (about 1100 measurements) to avoid averaging out the secular variation. The data span the past 5 m.y., a period that covers about 20 polarity changes and should therefore be long enough to provide a representative sample of the

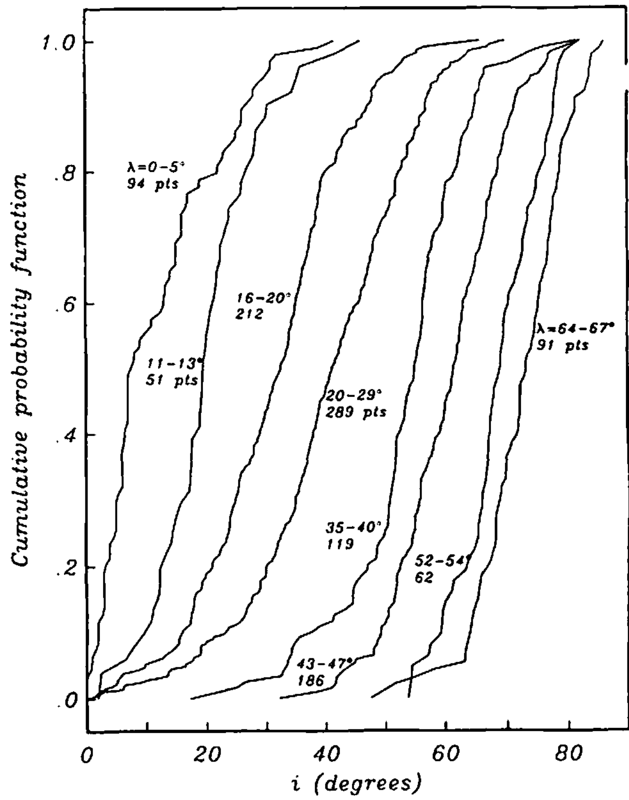

Fig. 3. Empirical cdfs for modified inclination i, using the available individual site data from *Lee* [1983]. All the data within each indicated latitude band are combined to generate the cdf.

secular variation but not so long that we have to take account of continental drift.

The quantities plotted are empirical cdfs for the absolute value of inclination of data from the latitude bands indicated. The original inclination data are all mapped onto the lower hemisphere in order to avoid the necessity of making a decision for each observation as to whether it arises from a normal or reversed main field. Correspondingly, the declinations (not shown) are mapped into the range $-90° < d \le 90°$. The use of the new variables is particularly important for low latitude sites, where secular variation might result in an apparent field reversal. Mathematically, we have used the following transformations from true declination D to d and from true inclination I to i:

$$i = |I|$$

and

$$d = \begin{cases} D, & -90° < D \le 90° \\ D + 180, & -180° < D \le -90° \\ D - 180, & 90° < D \le 180° \end{cases}$$

According to our nondipole model the northern and southern hemispheres are assumed to be symmetric in their behavior when averaged over the 5-m.y. period. Therefore data from equivalent latitudes in the two hemispheres can be combined to yield a better estimate for the cdfs. The symmetry means that any nonzero mean in the odd order terms in the spherical harmonic expansion can not be detected. The limitations of the data distribution are clearly visible in the jagged shapes of the cdfs, especially near the tails of the distribution.

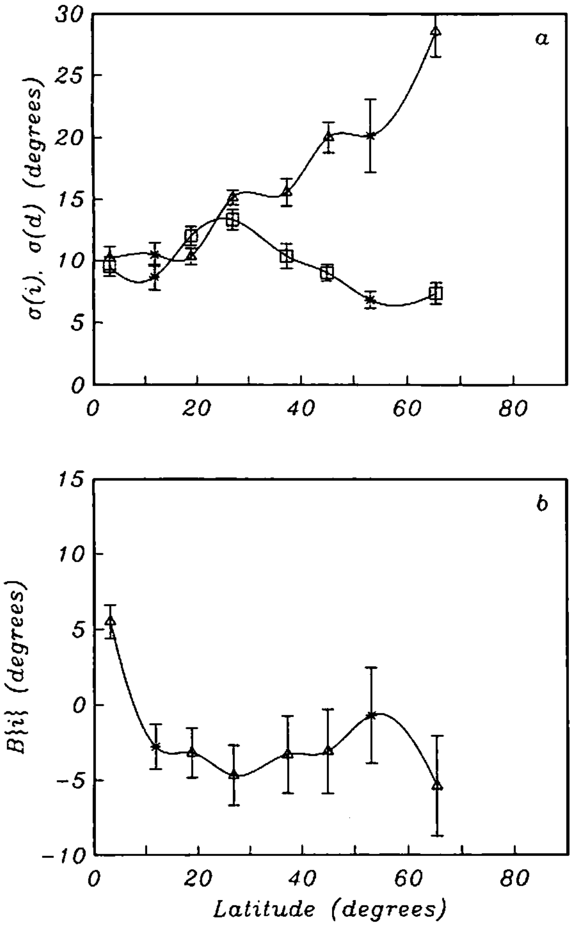

Fig. 4. Bias in i for the data grouped as in Figure 3 (Figure 4*b*). Data from the two smallest data groups are marked by a star to indicate their lesser reliability. Figure 4*a* shows the standard deviation in d (triangles) and i (squares) for the same data. Error bars are for one standard error computed by the bootstrap technique.

In order to quantify the variation in the cdfs we can estimate the means and standard deviation of the random variables d and i and examine them as functions of latitude. In Figure 4*a* are plotted the standard deviations, $\sigma(d)$ and $\sigma(i)$, respectively, for d (triangles) and i (squares) for each of the latitude bands, whose cdfs are given in Figure 3. Instead of the mean values of i we show the bias, $B\{i\}$, in Figure 4*b*. The bias is the expected value of i minus the value predicted for that latitude if the field were due to an axial geocentric dipole. The axial dipole predicts an inclination by the familiar formula

$$i_{ax} = \tan^{-1}(2\tan\lambda)$$

where λ is the latitude at the location in question. (Here the average latitude for the band, weighted by the number of data at each point, has been used.) Thus the bias is

$$B\{i\} = E\{i\} - i_{ax}$$

In order to obtain uncertainties for our estimates we used the bootstrap procedure [*Efron and Tibshirani*, 1986]. The error bars for the parameters in Figure 4 are for one standard deviation based on 500 bootstrap samples from each of the distributions shown in Figure 3. Naturally, the validity of the bootstrap estimates of standard error depends on how well the distribution functions of Figure 3 represent the range of secular variation.

The general picture that emerges for the parameters of the cdfs of Figure 3 is that the inclination standard deviation rises from about 10° at the equator to around 13° at a latitude of about 25° and then slowly decreases with increasing latitude. It should be noted that the standard deviation at the equator of i is one half that of the true inclination I. The declination standard deviation increases fairly steadily with latitude as does its uncertainty computed by the bootstrap. The solid curves connecting the data on Figure 3 are cubic spline interpolation between data points and serve merely to guide the eye. The bias in i is high near the equator (this is a consequence of mapping all the inclinations onto the lower hemisphere) but becomes consistently negative for latitudes higher than 7° or 8°. The size of the bootstrap error bars suggests that it would be unrealistic to interpret the detailed variations in bias with increasing latitude.

COMPARISON WITH THE SIMPLE MODEL

How do these data distributions compare with those computed from our giant Gaussian process (GGP)? Figure 5 shows the resulting pdfs for the GGP at a variety of latitudes for i and d. In these calculations we have simply set the mean of the magnitude of the axial dipole to today's value ($\bar{\gamma}_1^0 = 30\mu T$). The distribution for i is skewed at mid- and high latitudes, and the standard deviation is large at low latitudes. Figure 6 (solid curve) is a plot of the

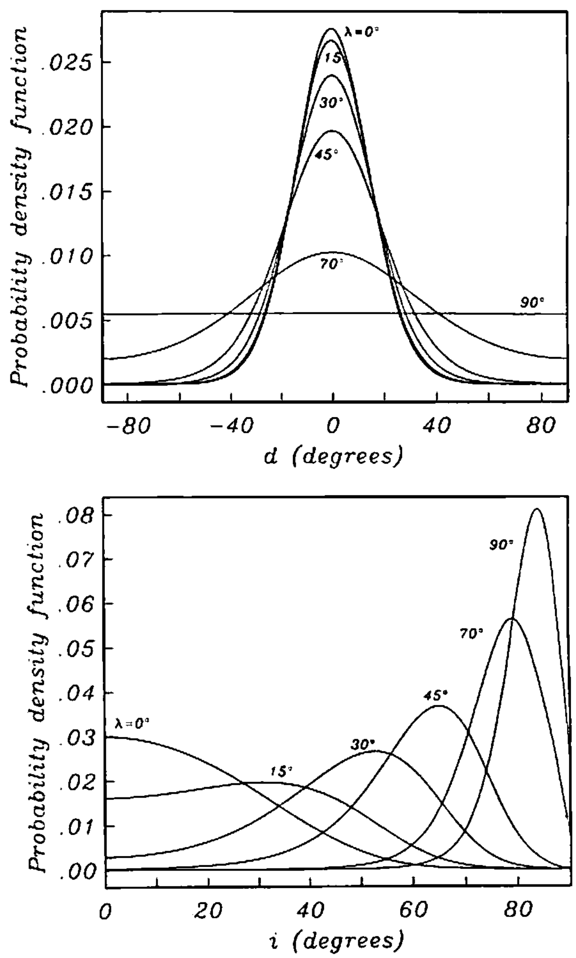

Fig. 5. Probability density functions for d and i at a variety of latitudes, assuming the giant Gaussian model for the secular variation, with a mean axial dipole corresponding to the present-day value.

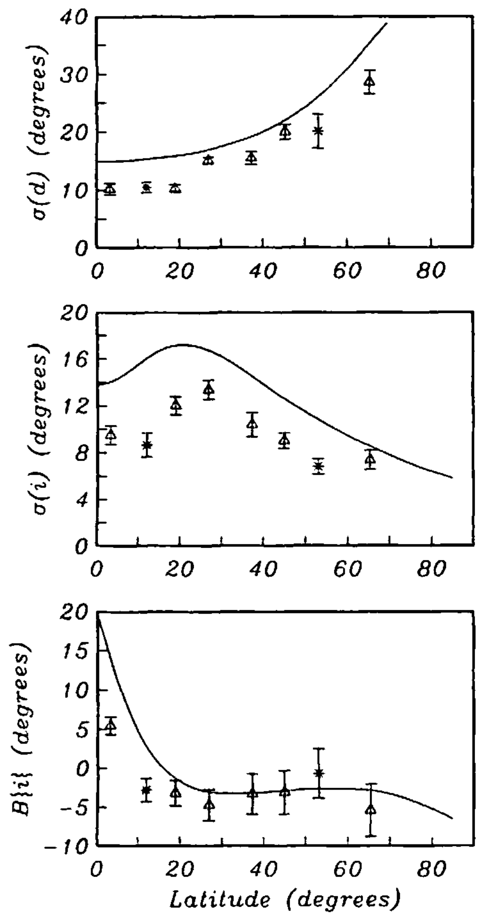

Fig. 6. Bias and standard deviations as a function of latitude for i and d for the model (solid curve) whose pdfs are shown in Figure 8 and the data of Figure 4 (symbols).

standard deviations and the bias as a function of latitude. Even though every Gauss coefficient except the one for the axial dipole has zero mean, this does not imply that i or even I must average to the axial dipole value at every latitude. In fact, there is a considerable body of paleomagnetic evidence in favor of biased inclinations (see, for example, *Wilson* [1971] and *Merrill and McElhinny* [1983 chapter 6]). Also plotted on Figure 6, as symbols, are the standard deviations and bias obtained for the global data compilation of *Lee* [1983] from Figure 4. It is evident that the model is unsatisfactory. The standard deviations are all too high, although the variation with latitude has approximately the right shape. The bias in i changes sign at too high a latitude (again partly reflecting the high standard deviation) and does not reach quite such low values in mid-latitudes as it does for the data.

How closely should we expect the data to follow the model? The error bars in Figure 6 are based on the assumption that the empirical distribution functions of Figure 3 represent the true distribution functions for the secular variation. The effects of the inadequacy of this assumption are hard to measure. Both the i and d data should have slightly higher standard deviations, $\sigma(i)$ and $\sigma(d)$, than predicted by the model because of the presence of errors in recording and measuring the field. For the data, $\sigma(i)$ will also be biased toward high values because inclinations from a range of latitudes with slightly differing means are combined to provide a single empirical distribution for that latitude band. The only mechanism by which the data standard deviation could be lower than that for the model is if there are insufficient data to sample the complete range of directions possible for the secular variation. *Constable* [1987] dicusses this problem and suggests that this influences one or two of the standard deviation points of Figure 4. These are marked by asterisks in Figure 6. The net result of the above possible sources of bias is that on the whole we should expect $\sigma(i)$ and $\sigma(d)$ for the model to lie slightly below those obtained for the data, except possibly for the data represented by the asterisks. From Figure 6 we must conclude that the present model is not in accord with the paleomagnetic data.

REFINING THE MODEL

We now seek the model with the least departure from isotropic Gaussian nondipole behavior and nonzero mean axial dipole. It is worth restating the assumptions for the original model with a view to adjustment of its basic parameters. The fundamental assumption is that the spatial statistics of the present-day field are typical of the paleofield and may be used to fix the variance α^2 for the scaled Gauss coefficients of the nondipole field as well as the nonaxial parts of the dipole field. The magnitude of the axial dipole is also taken to be Gaussian, with the same variance, and a mean value given by the present-day value for g_1^0. Three parameters of this model can be varied: (1) α, computed from the present-day field; it depends on the level in the Earth at which the spectrum is assumed to be flat; (2) $\bar{\gamma}_1^0$, the mean value for the axial dipole magnitude; it may not be the same as the present-day value; and (3) σ_1^2, the variance for the dipole components of the field; these may differ from the nondipole variance. If we depend on the paleomagnetic directional data, varying α and varying $\bar{\gamma}_1^0$ will have the same effect except that an increase in $\bar{\gamma}_1^0$ corresponds to a decrease in α. As we noted earlier, a decrease in α (resulting in a corresponding decrease in the variance of the field components) is not consistent with the known properties of the present field. In order to fit the modern field, *Langel and Estes* [1982] found a radius of $0.52a$ in place of our $0.547a$ for a white spectrum and $\alpha = 36.7$ instead of 27.7μT.

Another source of information is the limited available paleomagnetic intensity data for the past 5 m.y., which have been compiled by *McFadden and McElhinny* [1982]. They found from statistical analysis of virtual dipole moments that the average dipole moment over the past 5 m.y. is $8.67 \pm 0.65 \times 10^{22}$ A m^2 (to within the 95% confidence limit). This is within 10% of the present day value of 7.91×10^{22} A m^2. Experimentation with computing the model distribution parameters for various values of $\bar{\gamma}_1^0$ showed that it would require an increase of about 50% to reduce the standard deviations in i and d to about the right level. This is completely inconsistent with both the paleointensity data and the constraints on the present-day geomagnetic spectrum.

We conclude it would be difficult to make significant changes in either α or $\bar{\gamma}_1^0$. On the other hand, the constraints on σ_1 are much less rigorous. These come entirely from the present-day field values. Our Gaussian model for the field assumes that the scaled dipole coefficients γ_1^0, g_1^1, and h_1^1 are drawn from the same distribution as the rest of the coefficients (except that γ_1^0 has a nonzero mean). This is suggested by the fact that g_1^1 and h_1^1 do not alter the Kolmogorov-Smirnov test for normality when they are included in the distribution and both lie within one standard deviation of the mean. However, since only two numbers are involved, this fact is not statistically compelling. In any case,

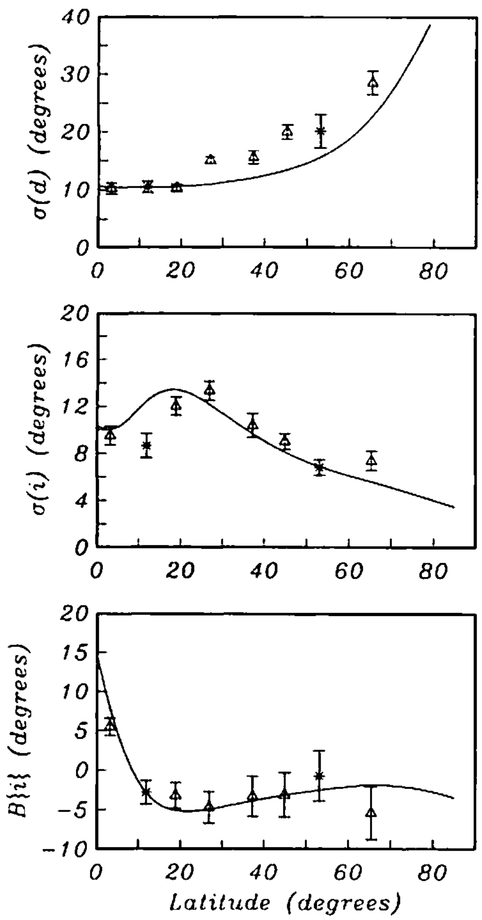

Fig. 7. Bias and standard deviations as a function of latitude for the preferred model described in the text. Parameters for the data compiled by *Lee* [1983] are again shown as open symbols.

there are good reasons to suspect that the dipole field is special. As can be seen in Figure 1, the dipole part of the spectrum lies substantially above the least squares line fit to a white source at the core. A reduction in σ_1, the variance of the dipole part of the field, will reduce the variances of i and d as desired. The present spectrum yields a value of $\sigma_1 = 0.207\bar{\gamma}_1^0 = 6.21\,\mu$T. We treat the dipole part of the field as a special case and reduce σ_1 to $0.1\bar{\gamma}_1^0 = 3.0\,\mu$T; this provides a quite acceptable fit to the standard deviation data of Figure 6. The present-day values of $g_1^1 = 0.0650 g_1^0$ and $h_1^1 = 0.1878 g_1^0$ are completely consistent with this new value, as both samples from the distribution would lie within 2 standard deviations of the zero mean.

The reduction of the variance in the directional data, by reducing σ_1 to $0.1\bar{\gamma}_1^0$ from $0.207\bar{\gamma}_1^0$, is obtained at the expense of almost eliminating the bias in i, but bias in this variable is a prominent feature in the observations. The zero crossing occurs at about the right latitude, but the large negative values at mid-latitudes cannot be obtained from a dipole field with such low variance. Another way of generating distributions with a significant bias in i is to include a zonal quadrupole term with a nonzero mean magnitude. The idea that the time-averaged geomagnetic dipole does not correspond to that of a geocentric axial dipole is not a new one: many paleomagnetists have proposed it (see, for example, *Wilson [1971]*, *Creer et al.* [1973], *Wells* [1973], *Georgi* [1974], *Wilson and McElhinny* [1974], *Merrill and McElhinny [1977]*, *Coupland and Van der Voo* [1980], and *Livermore et al.* [1983, 1984]). We find that the inclusion of a quadrupole term with a mean magnitude of 6% of that of the axial dipole but with the variance given by the spectrum as before provides a means of satisfying the paleomagnetic data. We assume that $\bar{g}_2^0$ is of the same sign as $\bar{g}_1^0$ and reverses sign with it. This is also consistent with the paleomagnetic evidence; it has been shown that the inclination bias is negative for both normal and reversed fields (see, for example, *Merrill and McElhinny* [1983, p. 185 figure 6.7]). Furthermore, the mean magnitude of our quadrupole term is comparable to the value estimated by other workers: for example, *Livermore et al.* [1983] give $\bar{\gamma}_2^0 = 0.05\bar{\gamma}_1^0$, in agreement with *Merrill and McElhinny* [1977].

Our final model is specified by the parameters $\alpha = 27.7$, $\bar{\gamma}_1^0 = 30$, $\bar{\gamma}_2^0 = 1.8$, and $\sigma_1 = 3.0\,\mu$T. The standard deviation and bias functions are shown in Figure 7. Figures 8 and 9 compare the cdfs for the preferred model with data distribution functions obtained from the global data. Here, the solid curves represent the average data cdfs over the latitude bands (0°–15°, 15°–30°, 30°–45° and 45°–70°) and the short dashed curves give the model cdfs for the boundaries of the latitude strips. These should represent bounds between which the data lie, provided the data are sufficiently good and the model is an adequate description of the statistics. For i (Figure 8) the average model expected from combining data at the given sites is computed and is given by the long dashed line. It agrees well with the empirical cdfs, except in the 0°–15° strip for which either a larger quadrupole term or a lower dipole standard deviation would improve the fit somewhat (but at the expense of the fit in the other bands). The declination data (Figure 9), as might be expected, are satisfied somewhat less well. This is mostly because in both the 15°–30° and 45°–70° strips the declinations are biased toward positive values, which could be due to bias arising from insufficient averaging over longitude in the sites available. Right-handedness of VGP positions has also been

TABLE 1. Parameters of Preferred Secular Variation Model

Parameter	Value
Standard deviation in spherical harmonic coefficients, μT	
σ_1	3.00
σ_2	2.14
σ_3	0.86
σ_4	0.37
$\sigma_l \quad l \ge 2$	$\left[\dfrac{(c/a)^{2l}\alpha^2}{(l+1)(2l+1)}\right]^{1/2}$
Mean values for the spherical harmonic coefficients, μT	
g_1^0	30.0
g_2^0	1.8
g_3^0	0.0
g_4^0	0.0
g_l^0	0.0
Standard deviation in surface field components, μT	
σ_θ	5.39
σ_ϕ	5.39
σ_r	9.68
Standard deviation in nondipole part of field components, μT	
$\hat{\sigma}_\theta$	4.48
σ_ϕ	4.48
σ_r	7.60

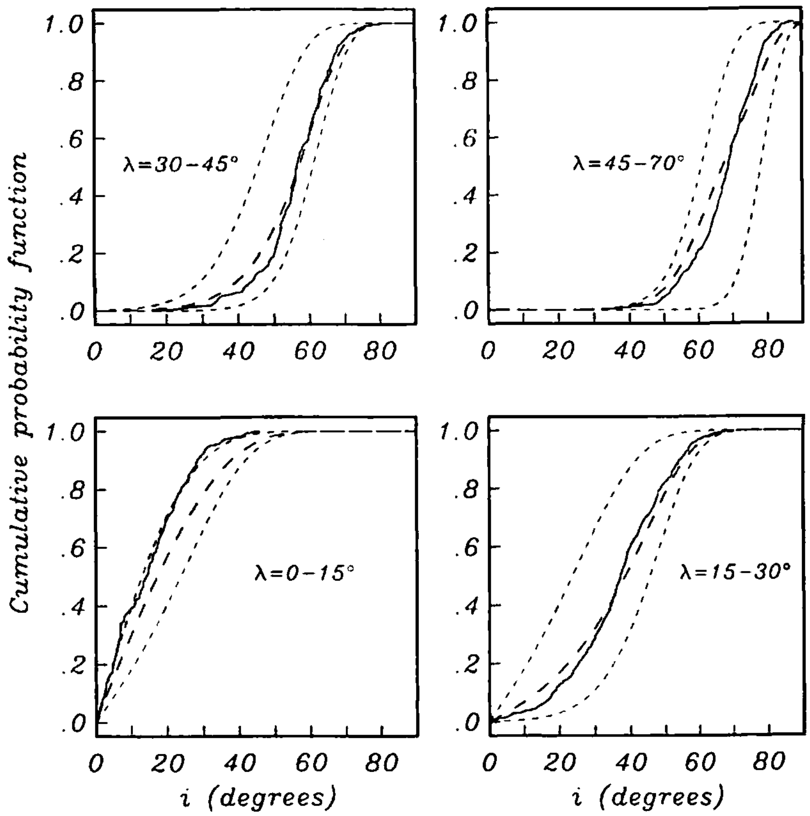

Fig. 8. The cdfs for *i* data of *Lee* [1983] grouped by indicated latitude bands (solid curve) compared with the predictions of the preferred model. Long dashed curves indicate the expected average cdf from combining data at the given sites. The short dashed curves indicate the region within which the observed cdf should lie.

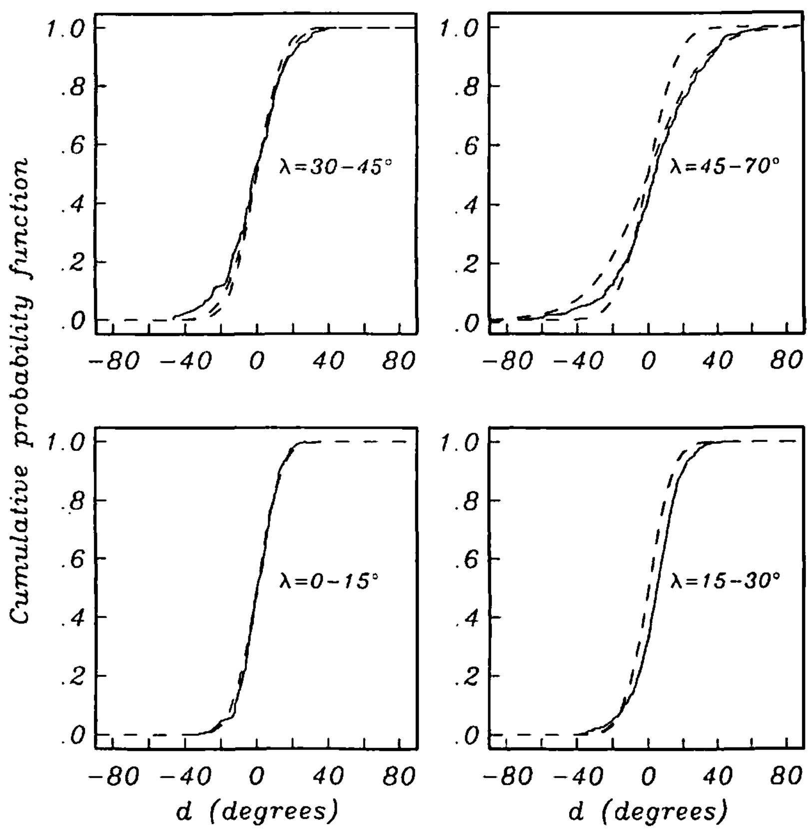

Fig. 9. The cdfs for *d* data of *Lee* [1983] grouped by indicated latitude bands (solid curve) compared with the predictions of the preferred model. Dashed curves indicate the region within which the observed cdf should lie.

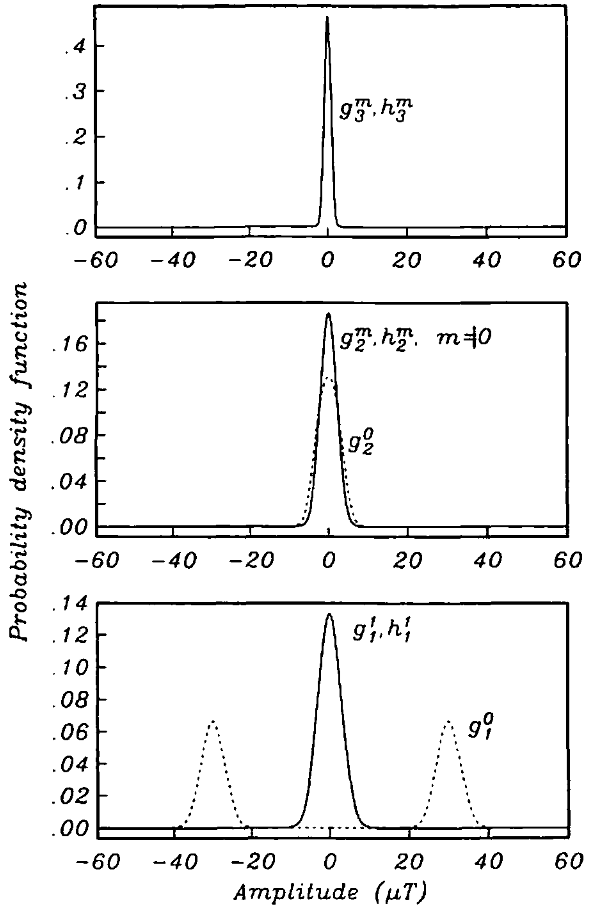

Fig. 10. Statistical distribution for the first 3 degree spherical harmonic coefficients for the model. Parameters for these distributions are given in Table 1.

noted by *Wilson* [1972] and others since then but *Lee* [1983] finds no evidence for believing the time averaged g_1^1 and h_1^1 terms significantly different from zero, when all the sources of error in the data are taken into account.

CONCLUSIONS

The statistical model described here is primarily interesting because of its simplicity and its ability to describe many of the observed features of the geomagnetic field despite the small number of free parameters involved. The primary features of the model are as follows.

1. With the exception of the axial dipole and axial quadrupole terms, each Gauss coefficient may be regarded as independently normally distributed with zero mean and a standard deviation for the nondipole terms consistent with a white source at the core surface.

2. The dipole part of the field does not have the same statistical distribution as the nondipole part. The axial dipole has predominated, and both the nonaxial parts of the dipole and the magnitude of the axial part of the dipole have lower variance (by about a factor of 4) than if they belonged to the same giant Gaussian distribution as the nondipole part of the field.

3. The hypothesis based on the present-day field that all the nondipole Gauss coefficients have zero mean is inconsistent with the paleodata. Although the inclination measurements are biased toward low values (relative to the geocentric axial dipole hypothesis) by the statistical variation in the Gauss coefficients, this bias alone is not sufficient to account for the inclination anomalies found in the paleodata; however, the inclusion of a quadrupole term with nonzero mean magnitude ($\bar{\gamma}_2^0 = 0.06\bar{\gamma}_1^0$ for the past 5 m.y.) provides a reasonably good fit to the paleodata and agrees with typical values found by other workers [*Livermore et al.*, 1983; *Lee*, 1983].

Thus the model may be described by just four free parameters. These are (1) the variance α^2 of the giant Gaussian process, which generates σ_l^2 the variance of each order term of degree l, (2) σ_1^2, the variance of each of the dipole terms, (3) $\bar{\gamma}_1^0$, the mean magnitude of the dipole field, and (4) $\bar{\gamma}_2^0$ the mean magnitude of the quadrupole field. These parameters are listed in Table 1.

The statistical distributions for the first 3 degree spherical harmonic coefficients are plotted in Figure 10 for this model. Note the quite narrow distribution of g_1^0 about its peak normal and reversed values, and how the variance drops off rapidly with increasing degree. The g_2^0 distribution results from the sum of two normal distributions, centered at $\pm 0.06\bar{g}_1^0$. There is an implicit covariance between $\bar{g}_2^0$ and $\bar{g}_1^0$, in that they are required to always have the same sign. The model for the Gauss coefficients also enables us to compute the probability density functions for any of the conventional field elements. In particular, the field components B_r, B_θ, and B_ϕ are easily shown to be Gaussian in

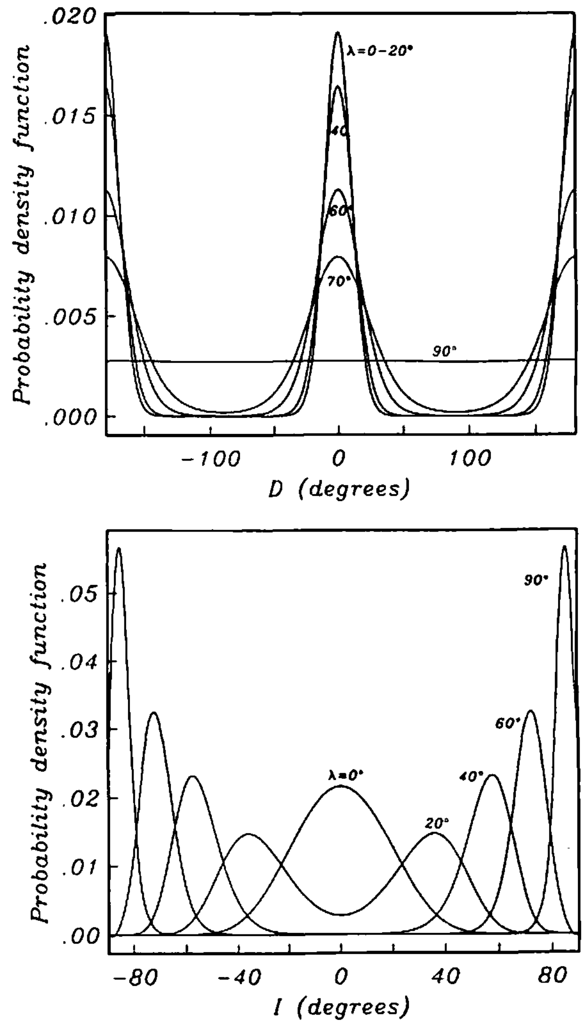

Fig. 11. Probability distribution functions for declination D and inclination I at a variety of latitudes for the preferred model.

distribution, with the horizontal variances locally equal but different from the variance in the vertical component (see Table 1). The pdfs for the traditional declination D and inclination I are shown in Figure 11. The distributions of other observable quantities can readily be computed. We plan to perform those calculations for some of the common paleomagnetic indicators (such as VGP dispersion) to subject the model to further scrutiny.

APPENDIX 1: SPHERICAL HARMONIC REPRESENTATION OF THE GEOMAGNETIC FIELD

$$\Psi = a\sum_{l=1}^{\infty}\sum_{m=0}^{l}\left(\frac{a}{r}\right)^{l+1}(g_l^m\cos m\phi+h_l^m\sin m\phi)P_l^m(\cos\theta)$$

$$\mathbf{B} = -\nabla\Psi$$

i.e.,

$$B_\theta = -\frac{1}{r}\frac{\partial\Psi}{\partial\theta} \quad B_\phi = -\frac{1}{r\sin\theta}\frac{\partial\Psi}{\partial\phi} \quad B_r = -\frac{\partial\Psi}{\partial r}$$

where g_l^m and h_l^m are the Schmidt partially normalized Gauss coefficients, a is the radius of the Earth, r, θ and ϕ are the usual spherical coordinates, and P_l^m are the partially normalized Schmidt functions related to the associated Legendre polynomials P_{lm} [*Abramowitz and Stegun*, 1965] by

$$P_l^m = P_{lm} \quad \text{for } m=0$$

$$P_l^m = \left|\frac{2(l-m)!}{(l+m)!}\right|^{1/2} P_{lm} \quad \text{for } m>0$$

The term "partial normalization" arises from the fact that within each degree l the average value of the square of P_l^m over the surface of the sphere has the same value for all values of m, i.e.,

$$\int_0^{2\pi}d\phi\int_{-1}^{1}d(\cos\theta)P_l^m(\cos\theta)\begin{Bmatrix}\cos m\phi\\ \sin m\phi\end{Bmatrix}P_{l'}^{m'}(\cos\theta)\begin{Bmatrix}\cos m'\phi\\ \sin m'\phi\end{Bmatrix} = \frac{4\pi}{2l+1}\delta_{ll'}\delta_{mm'}$$

A more convenient mathematical representation of the field is in terms of the fully normalized complex spherical harmonics, $Y_l^m(\theta,\phi)$, with complex coefficients b_l^m

$$\Psi = a\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(\frac{a}{r}\right)^{l+1}b_l^mY_l^m(\theta,\phi)$$

The fully normalized spherical harmonics Y_l^m are related to the associated Legendre polynomials $P_{lm}(\cos\theta)$ by

$$Y_l^m(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\,P_{lm}(\cos\theta)e^{im\phi}$$

and

$$\int_0^{2\pi}d\phi\int_{-1}^{1}d(\cos\theta)Y_{l'}^{m'*}(\theta,\phi)Y_l^m(\theta,\phi) = \delta_{ll'}\delta_{mm'}$$

Although the Schmidt normalization is commonly used in geomagnetism, we will use this fully normalized representation where it simplifies the mathematics (see Appendix 2). The fully normalized b_l^m are related to the Schmidt coefficients by

$$b_l^m = \begin{cases}(-1)^m\sqrt{\dfrac{2\pi}{2l+1}}\,[g_l^m-ih_l^m] & m>0\\[2ex] \sqrt{\dfrac{4\pi}{2l+1}}\,g_l^0 & m=0\\[2ex] (-1)^m(b_l^m)^* & m<0\end{cases}$$

APPENDIX 2: COMPUTATION OF VARIANCES AND COVARIANCES FOR B_r, B_θ, AND B_ϕ

As indicated in the section on nondipole field statistics, let us suppose that g_l^m and h_l^m are independent, normal random variables of zero mean for $l\geq2$. Then g_l^m and h_l^m satisfy the following:

$$E\{g_l^m\} = E\{h_l^m\} = 0$$

$$E\{g_l^mg_{l'}^{m'}\} = E\{h_l^mh_{l'}^{m'}\} = \sigma_l^2\delta_{ll'}\delta_{mm'}$$

$$E\{g_l^mh_{l'}^{m'}\} = 0 \qquad \text{for all } l,l',m,m'$$

where σ_l^2 is the variance of the normal distribution associated with degree l. First, let us derive the variance of the nondipole part of the field components B_r, B_θ, B_ϕ. This is facilitated by using the fully normalized spherical harmonic representation for the potential. Making use of the relationships between the b_l^m and the Schmidt coefficients, we find

$$E\{b_l^m(b_l^m)^*\} = \frac{2\pi}{2l+1}E\{(g_l^m)^2+(h_l^m)^2\} \qquad m\neq0$$

$$= \frac{4\pi}{2l+1}\sigma_l^2$$

$$E\{b_l^0(b_l^0)^*\} = \frac{4\pi}{2l+1}E\{(g_l^0)^2\}$$

$$= \frac{4\pi}{2l+1}\sigma_l^2$$

Thus $E\{b_l^m(b_l^m)^*\}$ is the same regardless of whether or not $m=0$. Because $E\{g_l^mh_{l'}^{m'}\}=0$ for $l=l'$ or $m=m'$ the only nonzero terms possibly remaining are of the form ($m\neq0$)

$$E\{b_l^m(b_l^{-m})^*\}$$

when g_l^m and h_l^m appear in both complex numbers.

$$E\{b_l^m(b_l^{-m})^*\} = (-1)^m\frac{2\pi}{2l+1}E\{(g_l^m)^2-(h_l^m)^2-2ig_l^mh_l^m\}$$

$$=0$$

We make use of the above expressions in deriving the field component variances and covariances. In addition, it is useful to remember the spherical harmonic addition theorem [*Jackson*, 1963, p. 67]:

$$\frac{2l+1}{4\pi}P_l(\hat{r}_0\cdot\hat{r}) = \sum_{m=-l}^{l}Y_l^m(\theta_0,\phi_0)Y_l^m(\theta,\phi)^*$$

Letting $\hat{r}=\hat{r}_0$ yields

$$\sum_{m=-l}^{l}|Y_l^m(\theta,\phi)|^2 = \frac{2l+1}{4\pi}$$

Differentiating this equation yields some further sums which are useful: $\partial/\partial\theta$ yields

$$\sum_{m=-l}^{l} Y_l^m(\theta,\phi)^* \frac{\partial Y_l^m}{\partial\theta} = 0$$

Some further manipulation and differentiation of the addition theorem yields

$$\sum_{m=-l}^{l} \left|\frac{\partial Y_l^{m*}}{\partial\theta}\right|^2 = \frac{l(l+1)(2l+1)}{8\pi}$$

We start with the radial field. From our nondipole field model, $E\{B_r\}=0$. Thus

$$\begin{aligned}\mathrm{Var}\{B_r\} &= E\{B_r B_r^*\}\\ &= \sum_l\sum_{l'}\sum_m\sum_{m'}(l+1)(l'+1)Y_l^m(Y_{l'}^{m'})^* E\{b_l^m b_{l'}^{m'*}\}\\ &= \sum_l (l+1)^2 \frac{4\pi\sigma_l^2}{2l+1}\sum_m |Y_l^m|^2\\ &= \sum_{l=2}^{\infty}(l+1)^2\left(\frac{c}{a}\right)^{2l}\frac{4\pi\alpha^2}{(l+1)(2l+1)^2}\frac{2l+1}{4\pi}\\ &= \alpha^2\sum_{l=2}^{\infty}\frac{l+1}{2l+1}\left(\frac{c}{a}\right)^{2l}\\ &= \alpha^2\left[\tfrac{1}{2}\left[1-\frac{c^2}{a^2}\right]^{-1} + \tfrac{1}{2}\left(\frac{a}{c}\right)\mathrm{th}^{-1}\left(\frac{c}{a}\right) - 1 - \frac{2}{3}\frac{c^2}{a^2}\right]\\ &= 0.0753\alpha^2 = (7.60)^2\,(\mu\mathrm{T})^2\end{aligned}$$

Similarly, $E\{B_\theta\}=0$ and

$$\begin{aligned}\mathrm{Var}\{B_\theta\} &= \sum_l \frac{4\pi\sigma_l^2}{2l+1}\sum_m\left|\frac{\partial Y_l^m}{\partial\theta}\right|^2\\ &= \sum_{l=2}^{\infty}\left(\frac{c}{a}\right)^{2l}\frac{4\pi\alpha^2}{(2l+1)^2(l+1)}\frac{l(l+1)(2l+1)}{8\pi}\\ &= \frac{\alpha^2}{2}\sum_{l=2}^{\infty}\frac{l}{(2l+1)}\left(\frac{c}{a}\right)^{2l}\\ &= 0.0262\alpha^2 = (4.48)^2(\mu\mathrm{T})^2\end{aligned}$$

Symmetry arguments clearly dictate that $E\{B_\phi\}=0$ and $\mathrm{Var}\{B_\phi\} = \mathrm{Var}\{B_\theta\}$.

Since B_r, B_θ, and B_ϕ are derived from sums of Gaussian random variables, they are clearly Gaussian also. Their covariances may be readily computed from

$$\begin{aligned}\mathrm{cov}\{B_r B_\theta\} = E\{B_r B_\theta^*\} &= \sum_l\sum_{l'}\sum_m\sum_{m'}(l+1)Y_l^m\left[\frac{\partial Y_{l'}^{m'}}{\partial\theta}\right]^* E\{b_l^m b_{l'}^{m'*}\}\\ &= \sum_{l=2}^{\infty}(l+1)\frac{4\pi\sigma_l^2}{2l+1}\sum_{m=-l}^{l}Y_l^m\left[\frac{\partial Y_l^m}{\partial\theta}\right]^*\\ &= 0\end{aligned}$$

Clearly, by symmetry $E\{B_r B_\phi^*\}=0$ and similarly for $E\{B_\theta B_\phi^*\}$. Thus the nondipole field components B_r, B_θ, and B_ϕ are independent Gaussian variables with zero mean and standard deviations which are equal for the two horizontal components and larger for the radial component.

APPENDIX 3: COMPUTATION OF THE DISTRIBUTION FUNCTIONS FOR DECLINATION AND INCLINATION

Let us suppose that the pdf for the axial part of the dipole field, f_D may be written as a sum of Gaussian kernel functions, $G_j(x)$, centered at positions X_j

$$f_D(x) = \sum_j c_j G_j(x)$$

where

$$G_j(x) = \exp -\tfrac{1}{2}\left\{\frac{(x-X_j)}{h_j}\right\}^2$$

and

$$\sum_j c_j h_j = \frac{1}{\sqrt{2\pi}} \qquad c_j = \exp\{\alpha_j\} \geq 0$$

This is a conventional technique of representing empirically an unknown pdf (see, for example, *Silverman* [1986]); $f_D(x)$ is known as a kernel estimator. We will assume that X_j, the locations of the kernels, and h_j, their widths, are chosen a priori and $h_j = h$ = const. Then the axial dipole contribution to the field components B_r and B_θ at any given site is distributed as

$$f_D(x') = \sum_j \frac{c_j}{k_i}\exp -\tfrac{1}{2}\left\{\frac{(x'-X_j')}{k_i h_j}\right\}^2$$

where

$$\begin{aligned} x' &= k_i x & k_i &= \cos\lambda \quad \text{for } B_\theta\\ X_j' &= k_i X_j & k_i &= \sin\lambda \quad \text{for } B_r\end{aligned}$$

λ is the latitude at the site in question.

In its simplest form (as is used in this work), $f_D(x)$ is just the sum of two Gaussian distributions, one centered on the value of the present-day normal field and the other on the reverse field (see the distribution function for g_1^0 in Figure 10). Then if, as discussed in the introduction, the nondipole part of the field has a Gaussian distribution $f_{ND}(y)$ that is independent of the dipole part of the field, the cumulative distribution function for the field components B_r and B_θ may be computed from

$$Pr(B_i \leq \beta) = \int_S f_D(x') f_{ND}(y)\,dx'dy$$

$$f_{ND}(y) = \frac{1}{\sqrt{2\pi}\sigma_i}\exp -\tfrac{1}{2}\left\{\frac{y}{\sigma_i}\right\}^2, \qquad \sigma_i = \sigma_\theta, \sigma_r, \sigma_\phi$$

where S is the region satisfying $x'+y \leq \beta$, and f_{ND} includes the g_1^1 and h_1^1 random variation. Then $f(\beta)$ the probability density function for the field component will be

$$f(\beta) = \frac{d}{d\beta}Pr(B_i \leq \beta)$$

and will again be a sum of Gaussians

$$f(\beta) = \sum_j \frac{c_j h}{(\sigma_i^2 + k_i^2 h^2)^{1/2}}\exp -\tfrac{1}{2}\left\{\frac{(\beta - X_j')^2}{(\sigma_i^2 + k_i^2 h^2)}\right\}$$

Note that this is easily modified to include a nonzero mean for the g_2^0 (and or any other higher order) term in the nondipole distribution function.

Now to compute the distribution function for $\tan I$, we need first to find that for $H = (B_\theta^2 + B_\phi^2)^{1/2}$. Letting

$$B_\theta = l\cos\theta \qquad B_\phi = l\sin\theta$$

and

$$Y_j = k_\theta X_j = X_j \cos\lambda$$

we have

$$Pr(H \le l) = \sum_j \frac{c_j h}{(\sigma_\theta^2 + k_\theta^2 h^2)^{1/2}} \int_0^l \int_0^{2\pi} \frac{l\,dl\,d\theta}{\sqrt{2\pi}\sigma_\phi} \cdot \exp -\tfrac{1}{2}\left\{ \frac{l^2 \sin^2\theta}{\sigma_\phi^2} - \frac{(l\cos\theta - Y_j)^2}{(\sigma_\theta^2 + k_\theta^2 h^2)} \right\}$$

or for the pdf for H

$$f_H(l) = \sum_j \frac{c_j h}{\sqrt{2\pi}\sigma_\phi(\sigma_\theta^2 + k_\theta^2 h^2)^{1/2}} \int_0^{2\pi} l\,d\theta \cdot \exp -\tfrac{1}{2}\left\{ \frac{l^2 \sin^2\theta}{\sigma_\phi^2} - \frac{(l\cos\theta - Y_j)^2}{(\sigma_\theta^2 + k_\theta^2 h^2)} \right\}$$

The pdf for $t = \tan I$ is given by

$$f_{\tan I}(t) = \int_0^\infty l\,dl\, f_r(tl) f_H(l)$$

where $f_r(tl)$ is the pdf for the radial component of the field at $R = H \tan I$, i.e.,

$$f_r(tl) = \sum_j \frac{c_j h}{(\sigma_r^2 + k_r^2 h^2)^{1/2}} \exp -\tfrac{1}{2}\left\{ \frac{(lt - Z_j')^2}{(\sigma_r^2 + k_r^2 h^2)} \right\}$$

and $Z_j = k_r X_j = 2X_j \sin\lambda$

Some algebraic manipulation yields

$$f_{\tan I} = C \sum_n c_n \sum_j c_j \exp(-\tau) \int_0^{2\pi} d\theta \left(\frac{\pi}{2\alpha^3}\right)^{1/2} \cdot \exp\left(\frac{\beta^2}{\alpha}\right) \operatorname{erfc}\left(-\frac{\beta}{\sqrt{\alpha}}\right)\left[\frac{\beta^2}{\alpha} + \frac{1}{2}\right] + \frac{\beta}{2\alpha^2}$$

with

$$\alpha = \alpha(\theta) = \frac{fg\sin^2\theta + eg\cos^2\theta + eft^2}{efg}$$

$$\beta = \beta(\theta) = \frac{g\cos\theta Y_j + ftZ_n}{fg}$$

$$\tau = \frac{gY_j^2 + fZ_n^2}{fg}$$

$$e = 2\sigma_\phi^2$$

$$f = 2(\sigma_\theta^2 + k_\theta^2 h^2)$$

$$g = 2(\sigma_r^2 + k_r^2 h^2)$$

$$C = \frac{h^2}{\sqrt{2\pi}\sigma_\phi(\sigma_\theta^2 + k_\theta^2 h^2)^{1/2}}$$

The θ integral can be computed rapidly using the trapezoidal rule. The inclination pdf is then

$$f_I(i) = \sec^2 i\, f_{\tan I}(t)$$

Similarly, the declination distribution may be computed from

$$Pr(d \le \delta) = \sum_j \frac{c_j h}{\sqrt{2\pi}\sigma_\phi(\sigma_\theta^2 + k_\theta^2 h^2)^{1/2}} \int_0^\delta \int_0^\infty l\,dl\,d\theta \cdot \exp -\tfrac{1}{2}\left\{ \frac{l^2 \sin^2\theta}{\sigma_\phi^2} - \frac{(l\cos\theta - Y_j)^2}{(\sigma_\theta^2 + k_\theta^2 h^2)} \right\}$$

and

$$f_D(\theta) = \sum_j \frac{c_j h}{\sqrt{2\pi}\sigma_\phi(\sigma_\theta^2 + k_\theta^2 h^2)^{1/2}} \int_0^\infty l\,dl \cdot \exp -\tfrac{1}{2}\left\{ \frac{l^2 \sin^2\theta}{\sigma_\phi^2} - \frac{(l\cos\theta - Y_j)^2}{(\sigma_\theta^2 + k_\theta^2 h^2)} \right\}$$

$$= \sum_j \frac{c_j h}{\sqrt{2\pi}\sigma_\phi(\sigma_\theta^2 + k_\theta^2 h^2)^{1/2}} \cdot \exp\left(\frac{\nu^2}{\mu} - \rho\right)\left[\left(\frac{\pi\nu}{4\mu^3}\right)^{1/2} \operatorname{erfc}\left(-\frac{\nu}{\sqrt{\mu}}\right) + \frac{1}{2\mu}\exp\left(-\frac{\nu^2}{\mu}\right)\right]$$

with

$$\mu = \mu(\theta) = \frac{\sin^2\theta(\sigma_\theta^2 + k_\theta^2 h^2) + \cos^2\theta\sigma_\phi^2}{2\sigma_\phi^2(\sigma_\theta^2 + k_\theta^2)}$$

$$\nu = \nu(\theta) = \frac{\cos\theta Y_j}{2(\sigma_\theta^2 + k_\theta^2 h^2)}$$

$$\rho = \frac{Y_j^2}{2(\sigma_\theta^2 + k_\theta^2 h^2)}$$

$\operatorname{erfc}(x)$ is the complete error function [*Abramowitz and Stegun*, 1965, p. 299]

The pdfs for I and D are readily converted to those for the modified field directions i and d. Comparison with the data requires cdfs for d and i. These are obtained by using a cubic spline quadrature scheme to integrate the numerical values of the pdfs obtained on a uniform grid for d and i. Obtaining the cdfs also provides some reassurance about the accuracy of the above results, since we can check that they integrate to unity.

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REFERENCES

Abramowitz, M., and I. A. Stegun, *Handbook of Mathematical Functions*, Dover, New York, 1965.

Baag, C., and C. E. Helsley, Geomagnetic secular variation model E, *J. Geophys. Res.*, *79*, 4918–4922, 1974.

Constable, C. G., Some statistical aspects of the geomagnetic field, Ph.D. thesis, Univ. of Calif., San Diego, La Jolla, 1987.

Coupland, D. H., and R. Van der Voo, Long-term nondipole components in the geomagnetic field during the last 130 m.y., *J. Geophys. Res.*, *85*, 3529–3548, 1980.

Cox, A., Analysis of the present geomagnetic field for comparison with paleomagnetic results, *J. Geomagn. Geoelectr.*, *13*, 101–112, 1962.

Cox, A., Latitude dependence of the angular dispersion of the geomagnetic field, *Geophys. J. R. Astron. Soc.*, *20*, 253–269, 1970.

Creer, K. M., The dispersion of the geomagnetic field due to secular variation and its determination for remote times from paleomagnetic data, *J. Geophys. Res.*, *67*, 3461–3476, 1962.

Creer, K. M., E. Irving, A. E. M. Nairn, Paleomagnetism of the Great Whin Sill, *Geophys. J. R. Astron. Soc.*, *2*, 306–323, 1959.

Creer, K. M., D. T. Georgi, and W. Lowrie, On the representation of the Quaternary and late Tertiary geomagnetic field in terms of dipoles and quadrupoles, *Geophys. J. R. Astron. Soc.*, *33*, 323–345, 1973.

Eckhardt, D. H., Correlations between global features of terrestrial fields, *Math. Geol.*, *16*, 155–171, 1984.

Efron, B., and R. Tibshirani, Bootstrap methods for standard errors, confidence intervals, and other measures of statistical accuracy, *Stat. Sci.*, *1*, 54–77, 1986.

Fisher, R. A., Dispersion on a sphere, *Proc. R. Soc. London, Ser. A*, *217*, 295–305, 1953.

Georgi, D. T., Spherical harmonic analysis of paleomagnetic inclination data, *Geophys. J. R. Astron. Soc.*, *39*, 71–86, 1974.
Gubbins, D., Geomagnetic field analysis, I, Stochastic inversion, *Geophys. J. R. Astron. Soc.*, *73*, 641–652, 1983.
Harrison, C. G. A., Secular variation and excursion of the earth's magnetic field, *J. Geophys. Res.*, *85*, 3511–3522, 1980.
Jackson, J. D., *Classical Electrodynamics*, John Wiley, New York, 1963.
Kendall, M., and A. Stuart, *The Advanced Theory of Statistics*, vol. 2, *Inference and Relationship*, 4th ed., MacMillan, New York, 1979.
Langel, R. A., and R. H. Estes, A Geomagnetic Field Spectrum; *Geophys. Res. Lett.*, *9*, 250–253, 1982.
Langel, R. A., R. H. Estes, G. D. Mead, E. B. Fabiano, and E. R. Lancaster, Initial geomagnetic field model form Magsat Vector data, *Geophys. Res. Lett.*, *7*, 793–796, 1980.
Lee, S., A study of the time-averaged paleomagnetic field for the past 195 million years, Ph. D. thesis, Aust. Nat. Univ., Canberra, 1983.
Livermore, R. A., F. J. Vine, and A. G. Smith, Plate motions and the geomagnetic field, I, Quaternary and late Tertiary, *Geophys. J. R. Astron. Soc.*, *73*, 153–171, 1983.
Livermore, R. A., F. J. Vine, and A. G. Smith, Plate motions and the geomagnetic field, II, Jurassic to Tertiary, *Geophys. J. R. Astron. Soc.*, *79*, 939–962, 1984.
Lowes, F. J., Spatial power spectrum of the main geomagnetic field and extrapolation to the core, *Geophys. J. R. Astron. Soc.*, *36*, 717–730, 1974.
Massey, F. J., The Kolmogorov-Smirnov test for goodness of fit, *J. Am. Stat. Assoc.*, *46*, 68–78, 1951.
Mauersberger, P., Das Mittel der Energiedichte des geomagnetishcen Hauptfeldes und der Erdoberfläche und seine säkulare Änderung, *Gerlands Beitr. Geophys.*, *65*, 207–215, 1956.
McElhinny, M. W., and R. T. Merrill, Geomagnetic secular variation over the past 5 m.y., *Rev. Geophys. Space Phys.*, *13*, 687–708, 1975.
McFadden, P. L., and M. W. McElhinny, Variations in the geomagnetic dipole, 2, Statistical analysis of VDMs for the past 5 million years, *J. Geomagn. Geoelectr.*, *34*, 163–189, 1982.
McFadden, P. L., and M. W. McElhinny, A physical model for paleosecular variation, *Geophys. J. R. Astron. Soc.*, *78*, 809–830, 1984.
Merrill, R. T., and M. W. McElhinny, Anomalies in the time-averaged paleomagnetic field and their implications for the lower mantle, *Rev. Geophys.*, *15*, 309–323, 1977.
Merrill, R. T., and M. W. McElhinny, *The Earth's Magnetic Field: Its History, Origin and Planetary Perspective*, Academic, San Diego, Calif., 1983.
Moffatt, H. K., *Magnetic Field Generation in Electrically Conducting Fluids*, Cambridge University Press, New York, 1978.
Silverman, B. W., *Density Estimation for Statistics and Data Analysis*, Chapman and Hall, New York, 1986.
Wells, J. M., Nonlinear spherical harmonic analysis of paleomagnetic data, in *Methods in Computational Physics*, vol. 13, *Geophysics*, edited by B.A. Bolt, pp. 239–269, Academic, San Diego, Calif., 1973.
Wilson, R. L., Dipole offset—The time-averaged paleomagnetic field over the past 25 million years, *Geophys. J. R. Astron. Soc.*, *22*, 491–504, 1971.
Wilson, R. L., Palaeomagnetic differences between normal and reversed field sources, and the problem of far-sided and right-handed pole positions, *Geophys. J. R. Astron. Soc.*, *28*, 295–304, 1972.
Wilson, R. L., and M. W. McElhinny, Investigation of the large-scale paleomagnetic field over the past 25m.y.; eastward shift of the Icelandic spreading ridge, *Geophys. J. R. Astron. Soc.*, *39*, 571–586, 1974.

C. G. Constable and R. L. Parker, Institute of Geophysics and Planetary Physics, Scripps Institution of Oceanography, University of California, San Diego, La Jolla, CA 92093.

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